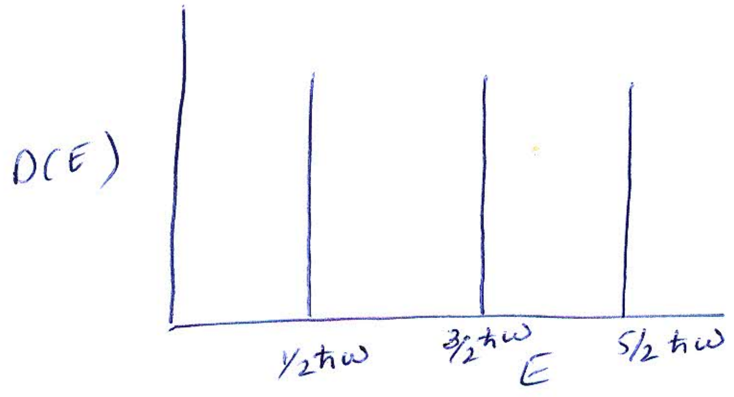
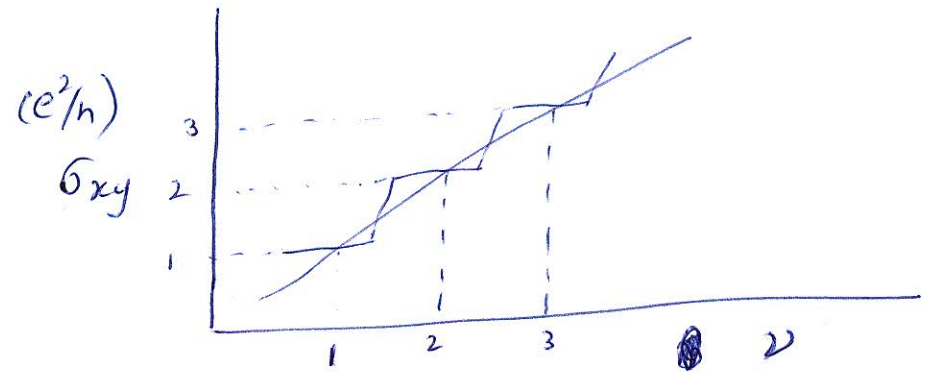
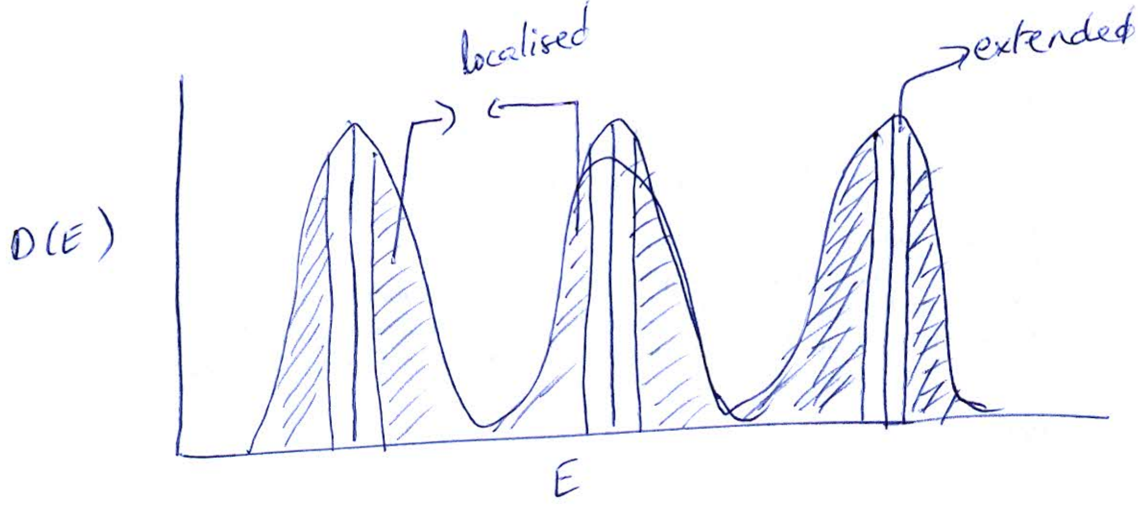
What we have shown is that the DOS ~~is~~ density of states is given by



We need to explain the plateaus in



But the D(E) plot only shows that as we change the # of electrons, the lowest LL gets occupied and then the next and so on. So according to this, the conductivity should go on increasing. To get the plateaux, we also need to include disorder. In the presence of disorder, the Landau levels get broadened into a band of energies with diff values. Most importantly, most of the states in the band get localised, except at the middle of the band.



Since current can only be carried by extended states, as the fillirp fraction is increased from 0, one first encounters localised states where not transport is possible, then when we encounter the first set of extended states in the middle of the first Landau band, the conductance rises abruptly ~~and~~ to e^2/h and then again remains flat, till it reaches the next set of extended states and jumps to $2e^2/h$ & so on. For the plateau to plateau transition to the be sharp, the # of extended states have to be very small. But those states have to carry the current that the full Landau level was carrying before disorder. So less # of electrons must be carrying more current, which means that they must be moving faster.

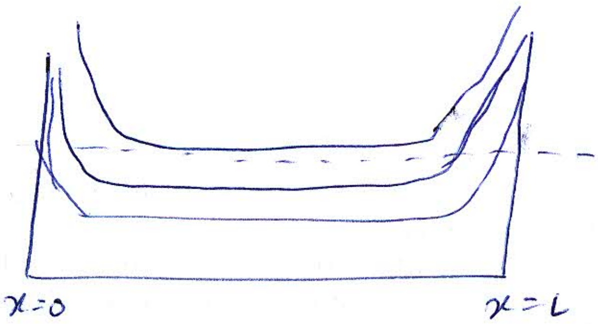
Note here, that we need disorder to get the plateaux, but also too much disorder will destroy IQHE. So an optimum amount of disorder is required.

I will not go through how one gets this kind of D(E). Instead, let me give ~~me~~ a ~~more~~ different ~~topological~~ explanation of why the quantisation is so accurate, in terms of edge states.

What happens at the edge of the sample? Essentially, at the edge of the sample the electrons feel a step potential that serves to keep them inside the sample.

~~There exist a semi-clanical way to understand why we get edge states~~

~~In the interior of the sample,~~

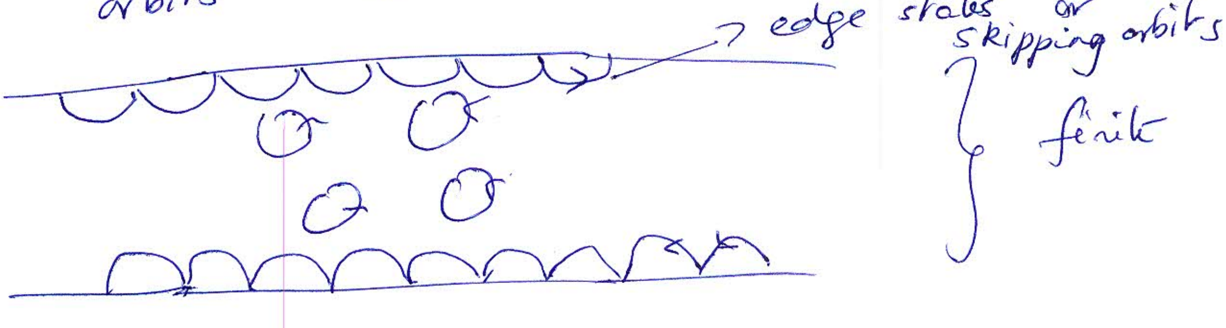


So even if we put our chemical potential in the gap, at the edges, it crosses the Fermi-level, so you have conducting edge states.

There is also a semi-clanical way to understand why we get edge states

In the interior of the sample, the Landau orbits are closed orbits (of the appropriate cyclotron radius)

But close to the edges, the electrons bounce off the boundaries and has skipping orbits and don't get localised



So due to the confining potential, the states at one edge of the sample typically go to the other edge. The electrons move in opposite directions at the 2 opposite edges.

Also, interior orbits are subject to the quantisation condition and can only occur for specific quantised energies.

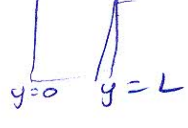
But the skipping orbits do not enclose any area (in the sense ~~they~~ [illegible] you normally quantise w.r.t area enclosed by the [illegible]) and hence they can occur for any energy.

Now, we can give a simple explanation for the QHE.

We solve the ~~[illegible]~~ problem of the ptcle in a magnetic field, but also with a confining potential $V(y)$ in the y-direction

So
$$H = \frac{\left(\vec{p} - \frac{e\vec{A}}{c}\right)^2}{2m} - eV(y)$$

(eg $V(y)$ can be a square ~~[illegible]~~ well)

or 

)

We don't need to solve it explicitly. All we need to use is that y is a fn. of p_x, which we have already found. In fact y ~~is a fn~~ increases monotonically with p_x. Let us now consider a full Landau level, — i.e. p_x in the range of $p_- < p_x < p_+$ which is set by $y(p_-) < y < y(p_+)$ where where $y(p_+) - y(p_-)$ = Width of the sample.

We will compute the current due to all these electrons:

$$I_x = \int \overbrace{\vec{J}(\vec{r})}^{\text{current density}} \cdot \hat{x} \, dy$$

$$= \frac{1}{L} \int J_x \, dx \, dy$$

To get the second line, we use the fact that current does not depend on what x we choose. So we can choose any x, integrate over it and divide by the length in the ~~x-direct~~ $\hat{x}$-direction, which we choose to be L; the sample length in the $\hat{x}$-direction

So $I_x = -\frac{e}{L} \int d^2r \; \rho(\vec{r})\, v_x(\vec{r})$

For $\rho(\vec{r}) = \sum_j \delta^2(\vec{r} - \vec{r}_j)$, we get

$$I_x = -\frac{e}{L} \sum_j V_{jx}(\vec{r})$$

$\hookrightarrow$ x-component of the velocity of the j^{th} particle.

We could also have written, in mtm space,

$$I_x = -\frac{e}{L} \sum_{\vec{p}} V_{px}\, n_p$$

$$= -e \int \frac{dp_x}{2\pi\hbar} V_{px}$$

So for the full Landau level,

$$I_x = -e \int_{p_-}^{p_+} \frac{dp_x}{2\pi\hbar} \frac{1}{m}\left(p_x - \frac{eA_x}{c}\right)$$

$$= -e \int_{p_-}^{p_+} \frac{dp_x}{2\pi\hbar} \frac{\partial H}{\partial p_x}$$

$$= -e \int_{p_-}^{p_+} \frac{dp_x}{2\pi\hbar} \frac{\partial E}{\partial p_x}$$

because, we are interested in the expectation value.

$$\frac{e}{h}\int_{E_-} dE = -\frac{e}{h}(\mu_+ - \mu_-)$$

$$\mu_+ - \mu_- = -eV_H = \text{Hall voltage}$$

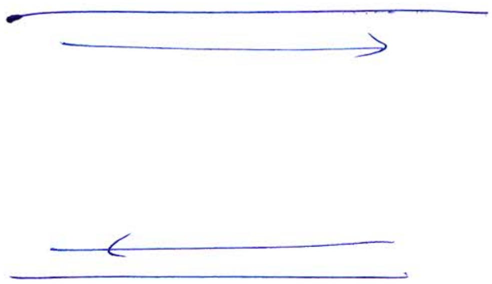
Hence we get

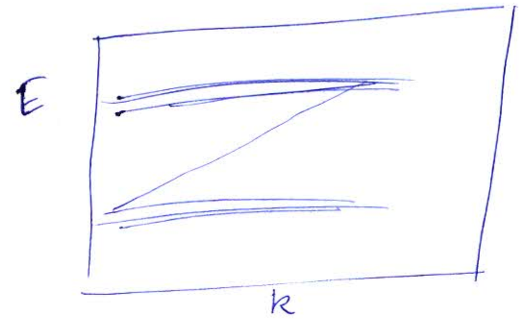
$$I = \frac{e^2}{h} V_H$$

Since this is true for each LL, we get

$$I = \frac{n e^2 V_H}{h}$$

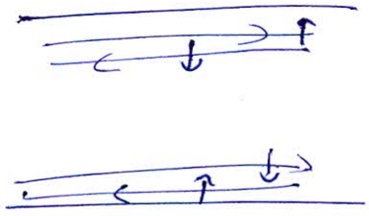
Here again, disorder can change single particle states in the interior of the sample. But as long as the strength of the impurity potential is small compared to the cyclotron energy, it cannot bind an electron. Another way of saying this is to see that the right-movers are at the top edge and the left-movers at the bottom edge. Hence they are spatially separated and one cannot have back-scattering unless the impurity potentials are strong enough to overcome this.

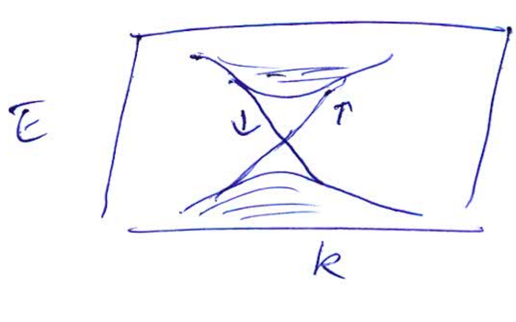




These are what are called chiral edge states. In the band structure of Landau levels, you can think of them as states crossing between 2 Landau levels. And the chirality is fixed by the sign of the magnetic field.

In fact, this is why people also call the IQHE system the first topological insulator. In what are now called topological insulators or quantum spin Hall insulators, there is no time-reversal breaking. Instead, one has spin ↑ going one way and spin ↓ going the other way



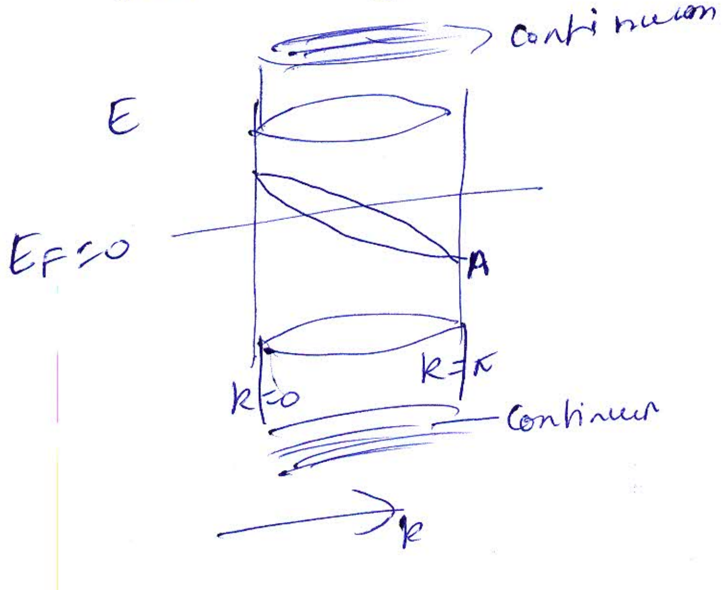


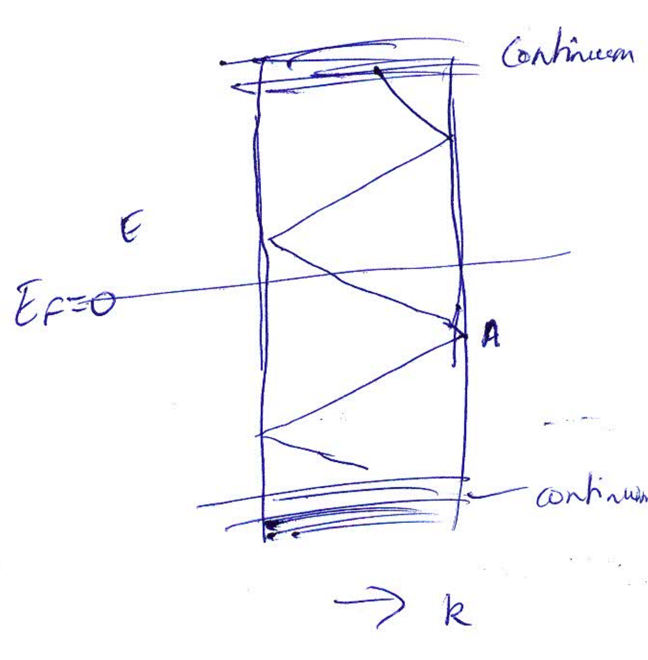
~~The~~ ~~But physically, the spin ↑ and spin ↓ don't always remain good quantum #'s in real systems~~

But when time-reversal symmetry remains unbroken, one can get topological insulators even when spin is not a good quantum #. This is a consequence of Kramer's theorem for spin 1/2 particles.

For a spin ½ system with time-reversal symmetry, all eigenstates have to be 2-fold degenerate - every state has to have a ~~degene~~ degenerate time-reversed partner state.

So if we plot the band structure of ~~conduction~~ insulators, we find that they fall into 2 classes





It is easy to see that these 2 classes (~~odd~~ even # of zero-crossings ~~versus~~ ~~[illegible]~~ versus odd # of zero-crossings) cannot be mapped into each other.

The 2 crossings can easily be made 0 crossings by moving the pt. A upwards. But 1 crossing remains 1 crossing, even if I move A upwards.

Kramer's theorem implies that at $k=0$ & $k=\pi$, the ~~sta~~ time-reversal invariant pts, the states have to be doubly degenerate, but ~~[illegible]~~ otherwise for other values of k, they don't. This gives us these 2 classes.

I will end this part, by saying that TI's (unlike QHE or superconductivity or many other effects) was one place where the theorist beat the exptalists in cond-mat. TI's were first predicted theoretically (in graphene, then in HgTe–CdTe quantum wells) and then exptally discovered in quantum wells and in 3D BiTe.